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Quantum Logical Description of Two-Particle Systems

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The quantum logical way of simulating quantum systems by automata is considered for two-particle systems. As an example, the EPR experiment with two spin-1/2 particles is considered and the violation of Bell's inequalities is demonstrated. Some methodological implications of the proposed approach are discussed.

1. INTRODUCTION

In our previous paper (Grib and Zapatrin, 1990) we described some simple one-particle systems by means of quantum logical non-Boolean lattices and graphs associated with them. Here we do the same for two-particle systems.

Two-particle systems are interesting because Jauch (1968) expressed a doubt about a "realistic" quantum logical interpretation applied to two particles. Following Finkelstein (1963), we shall say that properties of quantum systems do "exist," but that the logic differs from the usual human Boolean one. Then Jauch has put the question: what happens when we unite two quantum particles into a system? If the properties of individual particles "existed" before we united them, they must still "exist" in the system. However, we know that due to the complementarity of the whole and its parts, if the properties of the whole are observed as "existing," then we cannot say the same about its parts. And if the properties of the parts are supposed to exist, one comes to the validity of Bell's inequalities, which are broken for quantum systems (Grib, 1984).

In this paper we give the answer to Jauch's question. The point is that when we unite two particles into one quantum system we must add new elements to the property lattice, namely the questions corresponding to

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eigenstates of the permutation operator. This operator is nonlocal and it does not commute with local observables for parts of the system. Noncommutativity leads in turn to nondistributivity of the property lattice: properties of parts do exist in the system, but this is "nondistributive existence." Local observables have these OR those values AND nonlocal observables corresponding to the permutation operator have these OR those values; however, OR and AND are nondistributive: $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$.

In Grib and Zapatrin (1990) we have shown how one can simulate some simple quantum systems by classical automata with non-Boolean logics. This possibility of simulating quantum systems by classical automata described by graphs is important for two reasons. The first one is that it gives us a way to construct quantum computers—automata composed from classical elements but working analogously to quantum systems due to quantum logics. The second reason is that it has the profound meaning of showing why we can speak about quantum objects in terms of classical experiments. In some sense it corresponds to Lüdwig's (1989) extreme point of view that "atoms do not exist," and that only classical measuring apparata do really "exist," and "quantum objects" are merely a language describing relations between classical bodies and logics of these relations.

Nevertheless, we consider Lüdwig's interpretation extreme because classical bodies "consist" of quantum objects, but not the opposite, and macroscopic quantum mechanics gives us reason to believe that there are no purely classical objects.

After constructing graphs and property lattices for two-particle systems we give the rule for defining the wave function in terms of weights on graphs. Then we show how Bell's inequalities can be violated on graphs. This yields an example of Bell's inequalities breaking for classical systems with non-Boolean logics.

As we described in our previous paper, non-Boolean logics for classical systems can arise in situations when one uses negative logics, namely checking the state of a system through a negative answer on the opposite question. Such an approach is also developed in *linear logics* operating with *facts* which are properties satisfying the double negation rule: A is a fact if A is NOT NOT A.

At the end of the paper we discuss the problem of wave packet reduction for a two-particle system which as we think is connected with the Boolean nature of consciousness. Properties described by non-Boolean lattices do not correspond to events in Minkowski space-time and there is no usual probability attached to them. It is only due to "booleization" of the lattice done by an observer that they become events. This booleization is done by means of time, namely the observer can check values of noncommuting observables by measuring them at different moments of time. So an observer must move in time in order to apprehend through his Boolean consciousness the non-Boolean properties of a quantum system. We think this can help explain why we all move in time.

2. PRODUCT GRAPHS AND COUPLED SYSTEMS

First we briefly recall what we mean by the graph description of a quantum system S associated with Hilbert space $\mathcal{H}$. We select some properties (closed subspaces of $\mathcal{H}$) and associate with them vertices of a graph G. The edges of the graph G connect only the vertices associated with nonorthogonal subspaces. The obtained graph G is called the graph of the system S.

Now consider two quantum systems S_1 and S_2 associated with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and graphs G_1 and G_2, respectively. The Hilbert space of the compound system S is the tensor product $\mathcal{H}=\mathcal{H}_1 \otimes \mathcal{H}_2$. Building the graphs G_1 and G_2, we selected some subspaces in $\mathcal{H}_1$ and $\mathcal{H}_2$. All pairwise products of those subspaces generate the collection of subspaces of $\mathcal{H}$. To each subspace of this collection we associate a vertex of the product graph G; therefore, we define the set of vertices of the product graph G as the set of all ordered pairs of vertices of G_1 and G_2. The orthogonality on the set of such pairs is inherited from the lattice $\mathcal{L}(\mathcal{H})$, namely for i, $i' \in \mathcal{L}(\mathcal{H}_1)$ and k, $k' \in \mathcal{L}(\mathcal{H}_2)$

$$(i, k) \perp (i', k') \overset{\text{Def}}{\Leftrightarrow} i \perp i' \quad \text{or} \quad k \perp k'$$

Thus, the edges of the product graph connect two vertices (i, k) and (i', k') if both pairs of vertices i, i' of G_1 and k, k' of G_2 are connected by edges of G_1 and G_2, respectively.

Now consider two spin-1/2 particles and restrict possible spin measurements on the (xz) plane. This situation is described by the graphs of Figure 1 (Finkelstein and Finkelstein, 1982). Their property lattices are the simplest nondistributive ortholattices M_4 (see Figure 2).

The product graph $G = G_1 \otimes G_2$ has $4 \times 4 = 16$ vertices of the form ik, where $i = 1, 2, 3, 4$ runs over all vertices of G_1 and $k = 1, 2, 3, 4$ runs over

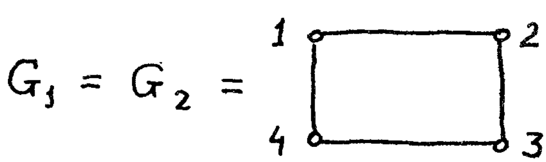

Fig. 1. The simplest graph with quantum property lattices.

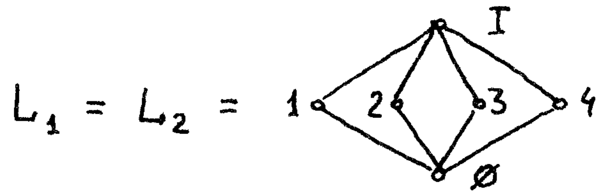

Fig. 2. Property lattices of G_1 and G_2.

all vertices of G_2. In accordance with the above definition, the graph G has the form given in Figure 3.

When a system S is represented by its graph G the (experimentally distinguishable) states of S are described by endowing the vertices of G with probability weights (interpreted as the possibility of occurrence of corresponding property). Let S be initially in a state $A=\{a_i\}$ described by a collection $\{a_i\}$ of probability weights on the vertices of G. Then the probability of finding S in a state $B=\{b_p\}$ is calculated by the transition probability formula [for a special case see Grib and Zapatrin (1990)]

$$P_{AB}=a_i T_i^p b_p + K \tag{2.1}$$

where summation by repeated indices is performed over all vertices of G. Here T_i^p is a symmetric matrix and K is a constant, both depending only on the form of the graph G. Denote the set of all states on G by $\mathscr{L}(G)$.

Here we consider the graph G (Figure 3) whose vertices are labeled by double indices. Since G is a product graph, we can consider two kinds of states. Factorizable states are represented by collections of vertex weights c_{ik} that can be represented as pairwise products of weights on G_1 and G_2.

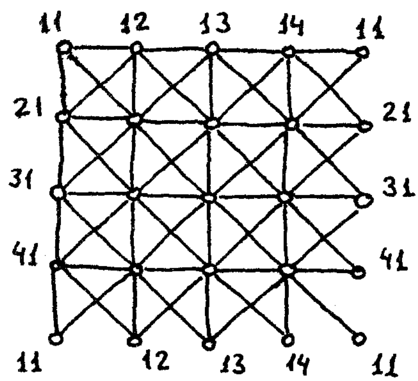

Fig. 3. The planar development of the product graph G (vertices labeled by the same indices are identical).

In other words, for $\{c_{ik}\} \in \mathscr{L}(G_1 \otimes G_2)$

$$(\{c_{ik}\} \text{ is factorizable}) \overset{\text{Def}}{\Leftrightarrow} \exists \{a_i\} \in \mathscr{L}(G_1) \\ \exists \{b_i\} \in \mathscr{L}(G_2) | c_{ik} = a_i b_k \tag{2.2}$$

If (2.2) does not hold for a state c_{ik}, it is called nonfactorizable.

The transition probability formula (2.1) for the graph G has the form

$$P_{CD} = c_{ik} T_{ik}^{pq} \, d_{pq} + K \tag{2.3}$$

where $K = 5/4$, and

$$T_{ik}^{pq} = \begin{cases} 1 & \text{if } \ p = i \text{ and } q = k \\ 0 & \text{if } \ p \perp i \text{ and } q \perp k \\ -1/4 & \text{otherwise} \end{cases}$$

[for graphs G_1, G_2 from Figure 1, $p \perp i$ means $p - i = 2 \pmod 4$].

The values of T_{ik}^{pq} and K can be obtained from the requirement of correspondence with the traditional quantum mechanical results. In general, given the matrices T_i^p, T_k^q and the constants k_1, k_2 for the graphs G_1 and G_2, the matrix T_{ik}^{pq} and the constant K for the product graph $G = G_1 \otimes G_2$ can be obtained from: (a) The requirement that both T_{ik}^{pq} and K do not depend on the values of weights (i.e., they are really constants); (b) the assumption that the two systems S_1 and S_2 are independent and thus for any pair of factorizable states $c_{ik} = a_i b_k$ and $d_{pq} = t_i s_q$ the transition probability is the product

$$P_{CD} = P_{AT} P_{BS}$$

where $C = \{c_{ik}\}, \ldots, S = \{s_q\}$

Now construct the property lattice $L(G)$ corresponding to the graph G (Figure 3). The maximal element $I \in L(G)$ is the 16-element set $V(G)$ of all vertices of the graph. The upper row has 16 elements of the form $\{i*, *k\}$, where $*$ runs over 1, 2, 3, 4. Each element of the upper row of $L(G)$ is a seven-element subset of the set $V(G)$ of all vertices of G. The next row downward consists of elements of three kinds: $\{i*\}$ and $\{*k\}$, which are four-element subsets of $V(G)$, and $\{ik, lm\}$ $(i \neq l, k \neq m)$, the two-element subsets. The direct calculation shows that there are four elements of the form $\{i*\}$ and $\{*k\}$ and 72 elements of the form $\{ik, lm\}$ $(i \neq l, k \neq m)$. The next row is the lowest. It consists of 16 elements which are one-element subsets $\{ik\}$ corresponding to each vertex $ik \in V(G)$. At the bottom of $L(G)$ is the void set $\varnothing$. Thus, we have completely described the property lattice $L(G)$ as the lattice of subsets of $V(G)$ partially ordered by set-theoretic inclusion.

The Hasse diagram of the lattice $L(G)$ consisting of $1 + 16 + 80 + 16 + 1 = 114$ elements is too complicated for typographical representation.

However, its graph representation is unambiguous. A mathematical treatment of questions concerning the graph representation of ortholattices is given in Zapatrin (1990*a*,*b*).

3. NONLOCAL QUESTIONS IN THE LATTICE AND GRAPH

The permutation operator has two eigenvalues: ± 1. To -1 corresponds the vector $<q-|$ in $\mathscr{H}$: $<q-| = 1/\sqrt{2}(\mathbf{e}_{12} - \mathbf{e}_{21})$; and to $+1$ corresponds its orthogonal complement.

First, we introduce into the graph G the new vertex corresponding to the following question for the system: is the wave function antisymmetric? The vertex $q-$ associated to this question is connected with other vertices of G according to the rule described in Section 2: two vertices are not connected by an arc if the subspaces associated to them are orthogonal; otherwise we draw an arc. Calculating directly the scalar products $\langle q-|ik\rangle$ we obtain that $\langle q-|$ is orthogonal only to vertices 11, 22, 33, and 44. Thus, the vertex $q-$ must be connected with all vertices of G except the above-mentioned ones. We can also introduce the vertex $q+$ associated with the subspace corresponding to $+1$. In this case we should connect q^+ with all vertices except $q-$. However, this vertex $q+$ will be redundant, namely the property lattice of this graph will be isomorphic to that of the graph H.

The graph H obtained from G by adding the vertex $q-$ is shown in Figure 4.

The property lattice $L(H)$ can be constructed from $L(G)$ described in Section 2 by adding two new elements. The first $\{q-\}$ is an atom (placed

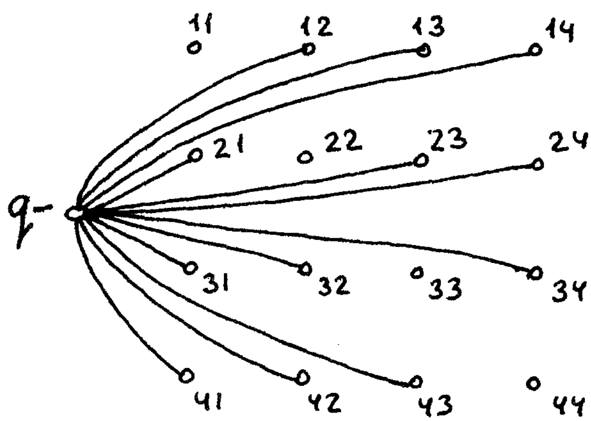

Fig. 4. The graph H. The old edges of G are omitted and only new ones are shown.

in the lowest row together with the $\{ik\}$). The other one, $\{11, 22, 33, 44\}$, is placed in the upper row. Also, the vertex $q-$ must be added to two-element subsets of the form $\{ik, ki\}$ of the middle row. This completes the description of the lattice $L(H)$.

4. BELL'S INEQUALITIES AND THEIR BREAKING IN TERMS OF GRAPHS

Let X, Y, Z be some elementary questions, and $\bar{X}$, $\bar{Y}$, $\bar{Z}$ their negations considered for a graph G. An elementary question means checking that one is in a state described by a collection of vertex weights $\{a_q\}$, where q runs over all vertices of the graph. The negation of the question $\{a_q\}$ means checking that one is in the state described by the collection $\{\bar{a}_q\}$, $\bar{a}_q = 1 - a_q$.

Then consider a compound system of two identical objects which is prepared in such a way that if we put one of the questions X, Y, Z to the first object and the same question to the second one, we always obtain exactly one YES and one NO. One can also put different questions to the objects when some answer on the question X to the first object is obtained, we immediately know the answer on this question to the second object (namely, the opposite answer). Then one could ask about the validity of Bell's inequality

$$P(X_1Y_1) + P(X_1Z_1) \geq P(Y_1Z_1)$$

However, none of the questions X_1X_1, X_1Z_1, and Y_1Z_1 can be put directly. So to convert Bell's inequality to measurable form, we equivalently have

$$P(X_1\bar{Y}_2) + P(X_1\bar{Z}_2) \geq P(Y_1\bar{Z}_2) \tag{4.1}$$

where, for example, $P(X_1\bar{Y}_2)$ is the probability to obtain YES for the question X to the first object and NO for the question Y to the second object.

Now let both objects be described by the graph G_1 (G_2) (Figure 1). This graph simulates spin measurements on a spin-1/2 particle restricted on the (xz) plane. Consider three elementary questions. Let $X =$ "$S_x = 1/2$?", $Y =$ "$S_\alpha = 1/2$?", and $Z =$ "$S_Z = 1/2$?", where α is an axis in the (xz) plane forming the angle α with the Z axis. These questions induce the following weights on the vertices 1, 2, 3, 4 of the graph G_1 (Grib and Zapatrin, 1990):

$$\begin{aligned} X:&\quad x_1 = x_3 = 1/2, \quad x_2 = 1, \quad x_4 = 0 \\ Y:&\quad y_1 = (1+\sin\alpha)/2, \quad y_2 = (1+\cos\alpha)/2, \quad y_3 = (1-\sin\alpha)/2, \\ &\quad y_4 = (1-\cos\alpha)/2 \\ Z:&\quad z_1 = 1, \quad z_2 = z_4 = 1/2, \quad z_3 = 0 \end{aligned} \tag{4.2}$$

For opposite questions we have $\bar{x}_i = 1 - x_i$, and so on.

As was proposed in Section 3, let our coupled system described by the graph G (Figure 3) be in the state $\{d_{pq}\}$—the eigenstate for the value -1 of the permutation operator. The state $\{d_{pq}\}$ is not an eigenstate for any question ik. Scalar products $|\langle q-|ik\rangle|^2$ yield

$$d_{pq}=\begin{vmatrix} 0 & 1/4 & 1/2 & 1/4 \\ 1/4 & 0 & 1/4 & 1/2 \\ 1/2 & 1/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 & 0 \end{vmatrix} \tag{4.3}$$

Form the product questions occurring in (4.1). All are factorizable and are calculated as pairwise products. For example, the collection of weights associated with the question $X_1 Y_2$ is $\{c_{ik}\}$, $c_{ik}=x_i(1-y_k)$, where the values of x_i and y_k are taken from (4.2).

The collection of weights associated with three questions from (4.1) is substituted into (2.2) to get the values of transition probabilities, which in our case are equal to

$$P(X_1 Y_2)=(1-\cos\alpha)/2$$

$$P(X_1 Z_2)=1/2$$

$$P(Y_1 Z_2)=(1+\sin\alpha)/2$$

If α is such that $1-\cos\alpha\geq\sin\alpha$, the inequality (4.1) is violated. We emphasize that the demonstrated violation of Bell's inequalities is essentially caused by the nondistributivity of the property lattice.

5. BOOLEIZATION THROUGH MEASUREMENT, THE ROLE OF CONSCIOUSNESS

One of the fundamental problems in quantum theory is the problem of measurement. London and Bauer (1939) discussed the idea that its solution is due to the special property of consciousness—its *introspection.* Introspection means knowledge as unambiguous identification of one's state of mind and leads to wave packet reduction, so that probabilities appear. Here we develop this idea further. We connect introspection with the Boolean logic of mind. Thus, if one considers the system: particle+apparatus+observer with mind, the wave packet reduction appears. This is because the Boolean-minded observer (being part of a non-Boolean system) projects the whole on his Boolean structure, which possesses the usual probability calculus. So it is the discrepancy between the non-Boolean structure of the world and the Boolean nature of mind that leads to wave packet reduction.

The *booleization* (projection of non-Boolean structure on its Boolean substructure) is made by means of time. For instance, one can have the idea that both time and movement in time are "invented" by the Boolean mind in order to grasp the non-Boolean nature of the world as well as the body with which this mind is intimately connected.

To understand this, consider as an example the system of two spin-1/2 particles and their spin projections on S_{z1} and S_{z2} on the z axis only. Construct the non-Boolean lattice of this system introducing the elementary questions corresponding to the following vectors in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\begin{aligned} \langle 1| &= e_1 \otimes e_1, \qquad \langle 2| = e_1 \otimes e_2 \\ \langle 3| &= 1/\sqrt{2}(e_1 \otimes e_2 - e_2 \otimes e_1) \\ \langle 4| &= e_2 \otimes e_1, \qquad \langle 5| = e_2 \otimes e_2 \end{aligned} \tag{5.1}$$

Szabó (1989), using "positive" reasoning, constructs a sublattice of $\mathcal{L}(\mathcal{H})$ generated by elements (subspaces) $1, \ldots, 5$. However, this lattice is not an ortholattice: it is impossible to define the orthocomplementation on Szabó's lattice; this fact can be proved exhaustively. However, we note that constructing the graph associated with the subspaces 1–5 in accordance with Section 2 (Figure 5a), we can see that the property lattice generated by this graph is isomorphic to the Boolean lattice 2^4 (generated by the graph of four disjoint vertices 1, 2, 4, 5) which essentially differs from Szabó's lattice (Figure 6a).

We need an ortholattice possessing the negative logic that we permanently apply (namely the identification of a property by checking its opposite). In order to obtain the ortholattice, we shall take not the sublattice of $\mathcal{L}(\mathcal{H})$ as Szabó (1989) does, but the subortholattice of the ortholattice $L(H)$ built in Section 3. The subortholattice constructed, call it M, is

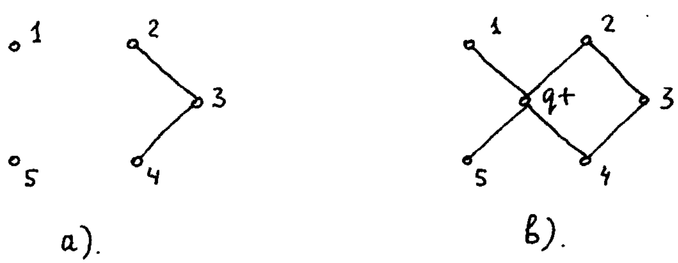

Fig. 5. (a) The graph associated with the collection (5.1). (b) The graph generating the ortholattice M.

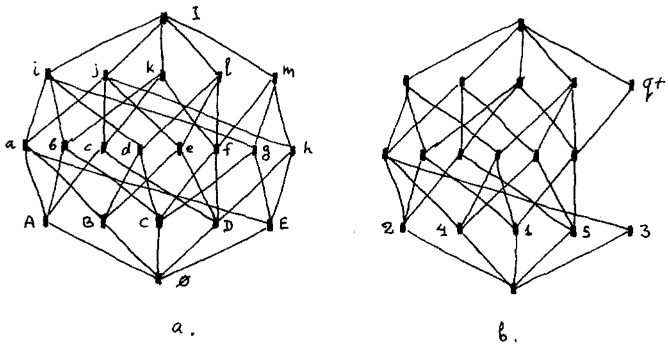

Fig. 6. (a) Szabó's model lattice. (b) The property ortholattice M (the vertex q^+ corresponds to the +1 eigenstate of the permutation operator). Note that the lattice of part (b) is obtained from part (a) by deleting elements g, h with adjacent links.

generated by the considered properties $1, 2, \ldots, 5$ (Figure 5b) which are associated with the following elements of $L(H)$:

$$1 \mapsto \{11\}, \quad 2 \mapsto \{13\}, \quad 3 \mapsto \{q-\}, \quad 4 \mapsto \{31\}, \quad 5 \mapsto \{33\}$$

The lattice M is constructed as the lattice 2^4 generated by elements 1, 2, 4, 5 with two additional elements 3 and $3^\perp$, which are connected with other elements as shown on Figure 6b. Due to the presence of these additional elements, M is non-Boolean; thus, one cannot define on it the usual probability measure; instead, we introduce some weights. For example, for a singlet state one can have the following collection of weights $\{w_i\}$:

$$w_1 = w_5 = 0, \qquad w_2 = w_4 = 1/2, \qquad w_3 = 1$$

Due to nondistributivity

$$2 = 2 \wedge (4 \vee 3) \neq (2 \wedge 4) \vee (2 \wedge 3) = 0 \vee 0 = 0$$

The property $1 \vee 2$ corresponds to the observation of $S_z^{(1)} = +1/2$ without the observation of anything for the second particle. The occurrence of $S_z^{(1)} = +1/2$ does not mean that 2 occurs, because w_i are not probabilities, and thus 2 cannot be called an event in the described experimental situation.

It is only if one "neglects" the element 3 that one obtains probabilities corresponding to a Boolean lattice. In order to "neglect" 3, the observer considers some other moment of time, so that 3 is now in the past and at the present moment only 1, 2, 4, 5 are actual. This corresponds to usual preparation and measurement procedures in quantum mechanics. Formally this can mean that we have a Hilbert space with a superselection rule associated with time. One can say that there are two Hilbert spaces parametrized by time moments t_1 and t_2 so that the performed measurement commutes with the permutation operator at moment t_1 and with the local operators $S_z^{(1)}$ and $S_z^{(2)}$ at moment t_2. The Boolean observer prepares the system at moment t_1 and obtains with probabilities 1/2 these or those values of $S_z^{(1)}$, $S_z^{(2)}$ at moment t_2.

6. SUMMARY

Two-particle quantum systems with spin can be simulated by classical automata described by graphs. These graphs are associated with nondistributive property lattices of these quantum systems. We emphasize that to nonlocal properties of a quantum system being in a certain eigenstate of the permutation operator there correspond merely some additional vertices in the graph which have nothing "nonlocal" in their nature. This leads to the possibility of violating Bell's inequalities in classical systems described by graphs (see Section 4) without breaking relativity theory.

The subjective interpretation of quantum mechanics of von Neumann, London, and Bauer can be connected with the Boolean nature of mind grasping the non-Boolean nature of the world which results in the projection postulate: the wave packet reduction. A simple example of it gives a two-particle system with spin.

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